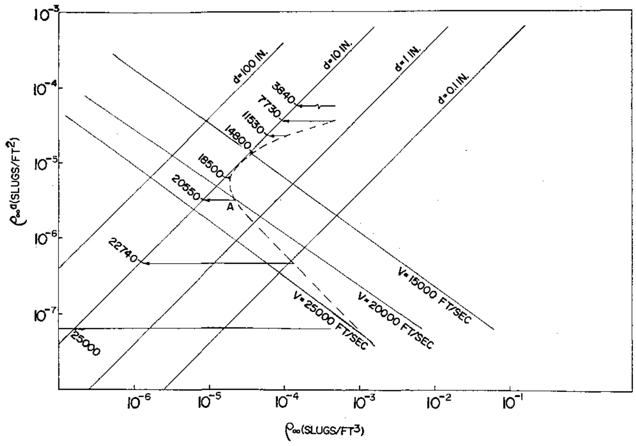

Fig. 2 Application to hypothetical re-entry trajectory. Hypothetical re-entry configuration assumptions: nose diameter d = 10 in., ballistic coefficient β = 100 lb/ft^2, entry velocity V_E = 25,000 fps, and entry angle γ_E = 17°. Values of velocity marked on trajectory are in units of fps; the broken line is locus of all points to left of which re-entry trajectory may be scaled (criterion: 0.01 on mass fractions at θ = 1 rad on ξ = 0.2 rad streamline).

It should be noted that the condition in the present method that mass fractions should agree to within 0.001 is a more stringent criterion than that adopted in Refs. 1 and 2. In fact Gibson and Marrone suggest that their criterion is equivalent to descrepancies in chemical concentration of about 10%, and should therefore be compared more properly with a criterion of 0.01 on mass fraction. The choice of criterion for binary scaling validity is of course arbitrary so that a direct comparison between the results of the Gibson and Marrone method and those of the present method is difficult. The apparent differences may be due to the inclusion of the $N_2 + O_2 \rightleftharpoons 2\ NO$ reaction in the present method, and the assumptions of Ref. 1 of 1) constant velocity along streamlines in the shock layer; 2) the equivalence of $\chi_{\rm crit}$ in blunt body and normal shock flows; and 3) the adoption of the nitrogen dissociation reaction as the controlling reaction for binary scaling. In connection with the last remark, it should be noted that in the present method the breakaway in chemical concentration as ρ_∞ was increased at a constant value of the product $\rho_\infty a$ did not occur with the same particular species for all $\rho_\infty a$ values considered.

Referring to Fig. 1, and taking the 0.01 on mass fractions criterion, it is seen for example that results for a 1-in.-diam model tested at a velocity of 20,000 fps and an ambient density of 4.8×10^{-5} slugs/ft^3 (corresponding to an ambient pressure of about 12-mm Hg) can be scaled directly to all sizes upwards in the direction of the arrow corresponding to the densities given by the abscissas in such a way that the product $\rho_\infty a$ is held constant (in this case = 2.0×10^{-6} slug/ft^2) for this fixed value of velocity.

Figure 2 shows the situation in terms of a hypothetical re-entry trajectory (ballistic coefficient β = 100 lb/ft^2, nose diameter = 10 in.). The dotted line indicates the locus of points that are the limits from which model results can be scaled. Thus point A in Fig. 2 indicates that the smallest model that can be used to simulate the full-scale conditions of 20,550 fps at 130,000 ft altitude ($\rho_\infty = 7.17 \times 10^{-6}$ slugs/ft^3) is one of 3.31-in. diam. The re-entry trajectory was computed according to the method of Allen and Eggers.[9]

It is demonstrated that binary scaling of model results to hypothetical re-entry sizes is feasible over limited ranges, but that full-scale testing is required to establish conditions over certain portions of trajectories.

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Large Deflection of Rectangular Sandwich Plates

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Nomenclature

a,b	= half length of panel in x and y directions
E	= elastic constant of the face layers
G_c	= elastic constant of the core layer
h	= thickness of the core layer
p	= external load per unit area
Q	= dimensionless loading parameter
t	= thickness of the face layer
u, v, w	= displacements in x, y, and z directions
U,V,W	= nondimensional parameter of displacements
w_0	= center deflection
W_0	= nondimensional center deflection ratio
α,β	= change of slope of the normal to the middle surface in x and y directions
θ	= ratio of h/a
λ	= ratio of a/b
μ	= ratio of t/a
ν	= Poisson's ratio of the core layer
ξ, η	= nondimensional parameters of x- and y- directional coordinates

The equations governing the large deflection of sandwich plates were derived by Reissner[1] using a simplified model consisting of two facings, acting as membranes, and a core resisting shear and normal stresses. These equations were verified by Wang,[2] employing the principle of complementary energy. Due to the appearance of the nonlinear terms, these equations become very complicated and do not lend themselves to a closed-form solution. The purpose of the present study is an attempt at obtaining an approximate solution for these equations by the method of successive approximation. The solution is based upon the smallness of central deflection ratio for the case of uniformly loaded, clamped, rectangular sandwich plates considering large deflections. This same

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method had been applied to homogeneous plates by Chien,[3] and the results agreed favorably with more exact solutions.

Consider a rectangular sandwich plate of $2a$ by $2b$ in the x and y directions respectively (Fig. 1) with a uniformly distributed load q per unit area. The basic displacement equations for a rectangular composite plate having a soft isotropic core and isotropic facings are[4]

$$\left.\begin{aligned}
&2\frac{\partial^2 u}{\partial x^2} + (1+\nu)\frac{\partial^2 u}{\partial y^2} + (1+\nu)\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \nu\left(\frac{\partial w}{\partial y}\right)^2\right] + (1-\nu) \times \left[\frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right] = 0 \\
&\frac{thE}{1-\nu^2}\left[2\frac{\partial^2 \alpha}{\partial x^2} + (1-\nu)\frac{\partial^2 \alpha}{\partial y^2} + (1+\nu)\frac{\partial^2 \beta}{\partial x \partial y}\right] - 4G_c\left(\frac{\partial w}{\partial x} + \alpha\right) = 0 \\
&p + hG_c\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right] + \frac{tE}{1-\nu^2}\left\{\frac{\partial^2 w}{\partial x^2}\left[2\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\right)^2 + \nu\left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{\partial^2 w}{\partial y^2}\left[2\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) + \left(\frac{\partial w}{\partial y}\right)^2 + \nu\left(\frac{\partial w}{\partial x}\right)^2\right] + 2(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \times \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right\} + \frac{thE}{2(1-\nu^2)} \times \\
&\quad \left\{\frac{\partial^2 w}{\partial x^2}\left[\frac{\partial \alpha}{\partial x} + \nu\frac{\partial \beta}{\partial y}\right] + \frac{\partial^2 w}{\partial y^2}\left[\frac{\partial \beta}{\partial y} + \nu\frac{\partial \alpha}{\partial x}\right] + (1-\nu)\frac{\partial^2 w}{\partial x \partial y}\left[\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\right]\right\} = 0
\end{aligned}\right\} \quad (1)$$

There are two additional equations in Eqs. (1). They are identical to the first two equations if x and y are interchanged, and u and v are interchanged in the first equation, α and β are interchanged, and x and y are interchanged in the second equation, and u, v, w, α, β, ν, E, G_c, t, and h denote, respectively, the displacements in the x, y, and z directions of points in the middle surface of the core, change of slope of the normal to the middle surface in x and y directions, Poisson's ratio and Young's modulus for the facings, shear modulus of the core, and the thickness of the facings and core.

In order to make the preceding equations dimensionless, introduce the following quantities:

$$\lambda = a/b \qquad \xi = x/a \qquad \eta = y/b$$

$$U = 12au/h^2 \qquad V = 12av/h^2 \qquad W = w/h$$

$$Q = 12a^3(1-\nu^2)p/th^2E \qquad \mu = t/a \qquad \theta = h/a$$

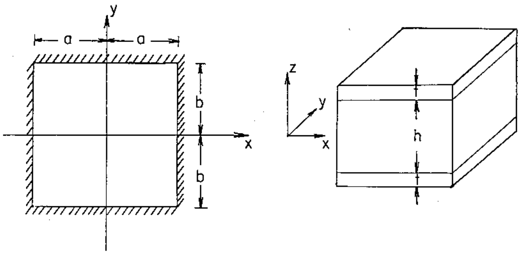

Fig. 1 Dimensions of rectangular sandwich plate.

Substituting into Eqs. (1) will yield

$$\left.\begin{aligned}
&2\frac{\partial^2 U}{\partial \xi^2} + (1-\nu)\lambda^2\frac{\partial^2 U}{\partial \eta^2} + (1+\nu)\lambda\frac{\partial^2 V}{\partial \xi \partial \eta} + 12\frac{\partial}{\partial \xi}\left[\left(\frac{\partial W}{\partial \xi}\right)^2 + \nu\lambda^2\left(\frac{\partial W}{\partial \eta}\right)^2\right] + 12(1-\nu)\lambda^2 \times \left[\frac{\partial^2 W}{\partial \xi \partial \eta}\frac{\partial W}{\partial \eta} + \frac{\partial^2 W}{\partial \eta^2}\frac{\partial W}{\partial \xi}\right] = 0 \\
&\frac{\mu\,\theta E}{1-\nu^2}\left[2\frac{\partial^2 \alpha}{\partial \xi^2} + (1+\nu)\lambda\frac{\partial^2 \beta}{\partial \xi \partial \eta} + (1-\nu)\lambda^2\frac{\partial^2 \alpha}{\partial \eta^2}\right] - 4G_c\alpha - 4\theta G_c\frac{\partial W}{\partial \xi} = 0 \\
&Q + \frac{\partial^2 W}{\partial \xi^2}\left\{2\theta\left(\frac{\partial U}{\partial \xi} + \nu\lambda\frac{\partial V}{\partial \eta}\right) + 6\left(\frac{\partial \alpha}{\partial \xi} + \nu\lambda\frac{\partial \beta}{\partial \eta}\right) + 12\theta\left[\left(\frac{\partial W}{\partial \xi}\right)^2 + \nu\lambda^2\left(\frac{\partial W}{\partial \eta}\right)^2\right]\right\} + \\
&\lambda^2\frac{\partial^2 W}{\partial \eta^2}\left\{2\theta\left(\lambda\frac{\partial V}{\partial \eta} + \nu\frac{\partial U}{\partial \xi}\right) + 6\left(\lambda\frac{\partial \beta}{\partial \eta} + \nu\frac{\partial \alpha}{\partial \xi}\right) + 12\theta\left[\lambda^2\left(\frac{\partial W}{\partial \eta}\right)^2 + \nu\left(\frac{\partial W}{\partial \xi}\right)^2\right]\right\} + \\
&(1-\nu)\lambda\frac{\partial^2 W}{\partial \xi \partial \eta}\left[2\theta\left(\lambda\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi}\right) + 6\left(\lambda\frac{\partial \alpha}{\partial \eta} + \frac{\partial \beta}{\partial \xi}\right) + 24\lambda\theta\frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\right] + \frac{12\,G_c}{\mu\theta}\,\frac{1-\nu^2}{E} \times \\
&\left(\frac{\partial \alpha}{\partial \xi} + \lambda\frac{\partial \beta}{\partial \eta}\right) + \frac{12G_c}{\mu}\,\frac{1-\nu^2}{E} \times \left(\frac{\partial^2 W}{\partial \xi^2} + \lambda^2\frac{\partial^2 W}{\partial \eta^2}\right) = 0
\end{aligned}\right\} \quad (2)$$

Again, two equations have been omitted; these may be obtained in the manner outlined previously.

Equations (2) are to be solved using the following boundary conditions:

$$\left.\begin{aligned}
W = U = V = 0 &\quad \text{at} \quad \xi = \pm 1 \text{ or } \eta = \pm 1 \\
\alpha = \beta = 0 &\quad \text{at} \quad \xi = \pm 1 \text{ or } \eta = \pm 1 \\
\partial W/\partial \xi = 0 &\quad \text{at} \quad \xi = \pm 1 \\
\partial W/\partial \eta = 0 &\quad \text{at} \quad \eta = \pm 1
\end{aligned}\right\} \quad (3)$$

If we assume the magnitude of w_0 is, at most, of the same order as h, we can define the center deflection ratio as $W_0 = w_0/h \leq 1$ and use it as a perturbation parameter. The solutions of the nondimensional Eqs. (2) may be expressed in ascending powers of W_0 n the following manner:

$$\left.\begin{aligned}
Q &= a_1W_0 + a_3W_0^3 + \ldots \\
W &= w_1(\xi,\eta)W_0 + w_3(\xi,\eta)W_0^3 + \ldots \\
U &= s_2(\xi,\eta)W_0^2 + s_4(\xi,\eta)W_0^4 + \ldots \\
V &= t_2(\xi,\eta)W_0^2 + t_4(\xi,\eta)W_0^4 + \ldots \\
\alpha &= m_1(\xi,\eta)W_0 + m_3(\xi,\eta)W_0^3 + \ldots \\
\beta &= n_1(\xi,\eta)W_0 + n_3(\xi,\eta)W_0^3 + \ldots
\end{aligned}\right\} \quad (4)$$

where, by definition, we require that

$$w_1(0,0) = 1 \qquad w_2(0,0) = w_3(0,0) = \ldots = 0 \quad (5)$$

w_i, s_i, t_i, m_i, and n_i, ($i = 1,2,3 \ldots$) are functions of ξ and η, and a_i is the constant coefficient.

Substituting Eqs. (4) into Eqs. (2) and using the boundary

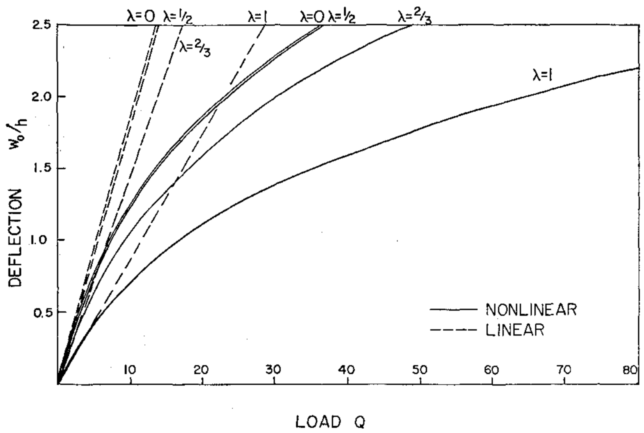

Fig. 2 Deflections of rectangular sandwich plates with the edges clamped.

conditions of Eqs. (3), and equating like powers of W_0, we obtain a series of linear differential equations and boundary conditions, i.e., for the first-order approximation,

$$\frac{\mu\theta E}{1-\nu^2}\left[2\frac{\partial^2 m_1}{\partial\xi^2} + (1+\nu)\lambda\frac{\partial^2 n_1}{\partial\xi\partial\eta} + (1-\nu)\lambda^2\frac{\partial^2 m_1}{\partial\eta^2}\right] - 4G_c m_1 - 4\theta\, G_c\frac{\partial w_1}{\partial\xi} = 0$$

$$a_1 + \frac{12G_c(1-\nu^2)}{\mu\theta E}\left[\left(\frac{\partial m_1}{\partial\xi} + \lambda\frac{\partial n_1}{\partial\eta}\right) + \theta\left(\frac{\partial^2 w_1}{\partial\xi^2} + \lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)\right] = 0 \qquad (6)$$

Here again there is one additional equation that can be obtained by interchanging m_1 and n_1, ξ and η in the first equation.

The corresponding boundary conditions are

$$w_1 = m_1 = n_1\ 0 \quad \text{at} \quad \xi = \pm 1 \quad \text{or} \quad \eta = \pm 1$$
$$\partial w_1/\partial\xi = 0 \quad \text{at} \quad \xi = \pm 1$$
$$\partial w_1/\partial\eta = 0 \quad \text{at} \quad \eta = \pm 1$$
$$w_1(0,0) = 1$$

Equations for higher-order approximations can be obtained in a similar manner.

The solutions of Eqs. (6) may be assumed in the form of polynomials in terms of ξ and η as

$$w_1 = (1-\xi^2)^2(1-\eta^2)^2(1 + B_1\xi^2 + C_1\eta^2 + D_1\xi^4 + E_1\eta^4 + F_1\xi^2 n^2)$$
$$m_1 = (1-\xi^2)(1-\eta^2)\xi(A_2 + B_2\xi^2 + C_2\eta^2 + D_2\xi^4 + E_2\eta^4 + F_2\xi^2\eta^2)$$
$$n_1 = (1-\xi^2)(1-\eta^2)\eta(A_3 + B_3\xi^2 + C_3\eta^2 + D_3\xi^4 + E_3\eta^4 + F_3\xi^2\eta^2)$$

The preceding expressions satisfy the boundary conditions Eqs. (7), identically.

Substituting these polynomials in Eq. (5), we determine the coefficients by equating like powers of ξ and η. By repeating this procedure, we may extend the method to any higher-order approximation.

As a numerical example, consider a rectangular sandwich plate with the following properties:

$$E = 10.5 \times 10^6 \text{ psi} \qquad G_c = 50{,}000 \text{ psi}$$
$$\nu = 0.3 \qquad \mu = 0.0006 \qquad \theta = 0.04$$

The numerical results for the example, based upon the perturbation method for the case $\lambda = 0, \frac{1}{2}, \frac{2}{3}$, and 1, are plotted in the Fig. 2.

For a square plate, the result of a two-terms approximation for the dimensionless loading as a function of the center deflection ratio is recorded as follows:

$$Q = (11.4074 w_0/h) + 5.2596(w_0/h)^3$$

For $w_0/h = 1$, the error in the linear theory is seen to be about 30%. No effort has been made to improve the accuracy of the numerical result since this can be done easily by increasing the number of terms in the expansion.

This method can also be used for plates with other boundary conditions by changing the assumed polynomials to satisfy the appropriate boundary conditions. For instance, for a simply supported plate, the deflection and bending moments must be zero along the boundaries. When free edges are considered, there will be no moments or shears at the edges.

It is seen from the figure that when $\lambda = \frac{1}{2}$, the curve practically coincides with the curve of $\lambda = 0$; therefore, for rectangular plates with an $a/b \leq \frac{1}{2}$, we can treat it as an infinitely long plate.

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Modification of Elements in the Displacement Method

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Introduction

In the analysis or design of most structures, it is usually necessary to alter parts of the original configuration and determine the resulting stresses and/or deflections. A case in point is the so-called "fail safe" analysis where the response is required after members have been systematically and progressively deleted from the structure. Obviously, only very few changes have been made in the structure during any one step, and it would be desirable not to be required to perform the analysis ab initio.

The purpose of this note is to explore the implications and ramifications of the theory of cutouts in the displacement method. In addition, a method for computing a modified flexibility matrix will be illustrated wherein the results from the original structure can be used and updated.

Modification of the Stiffness and Flexibility Matrices

The modified stiffness matrix obtained from the original stiffness matrix as discussed in Ref. 3 can be written as

$$\mathbf{K}_m = \mathbf{K} + \mathbf{a}_h{}^T\mathbf{C}^{-1}\,\delta\mathbf{k}_h\,\mathbf{a}_h \qquad (1)$$

where $\mathbf{K}_m$ = the modified reduced stiffness matrix, $\mathbf{K}$ = the original reduced stiffness matrix, $\mathbf{C} = [\mathbf{I} + \delta\mathbf{k}_h\,\mathbf{a}_{1h}\,\mathbf{K}_{11}{}^{-1}\,\mathbf{a}_{1h}{}^T]$, and $\mathbf{K}_{11}$ = the partition of the total merged stiffness

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